DRIVEN ANOMALOUS DYNAMICS BREAKING OF EINSTEIN RELATION AND SCALING PROPERTIES

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OUTLINE

Einstein relation for a Brownian particle

Continuous time random walk and anomalous diffusion

Breaking of the Einstein relation in superdiffusive model

The importance of Rare Events

Field-induced anomalous dynamics

Field-induced anomalous dynamics in a subdiffusive model

Conclusions

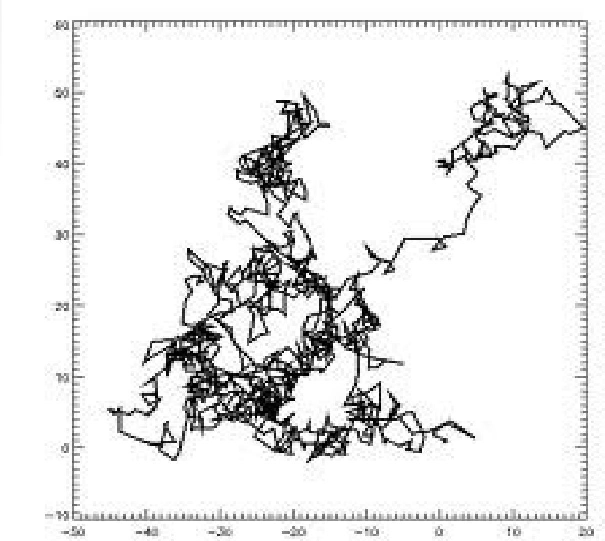
EINSTEIN RELATION FOR A BROWNIAN PARTICLE

$$m\dot{v} = -\gamma v + \sqrt{2\gamma T}\ \eta \qquad \langle \eta \rangle = 0$$

Colloidal particle immersed in an equilibrium fluid

Mean squared displacement with no external perturbation

$$\langle x^2(t) \rangle = \frac{2\ T}{\gamma} t = 2D\ t$$



$$m\dot{v} = \ldots + F \qquad \langle x(t) \rangle_F = \overline{\delta x(t)}_F = \mu F t$$

External drag field

$$\mu = \beta D$$

$$\frac{\langle x^2(t) \rangle}{\langle x(t) \rangle_F} = \frac{2}{\beta F}$$

Einstein relation

$$x_{i+1} = x_i + u$$

$$p(u) = \text{gaussian}$$

"OUT-OF-EQUILIBRIUM" EINSTEIN RELATION FOR A BROWNIAN PARTICLE (reference state with a current)

$$m\dot{v} = -\gamma v + \sqrt{2\gamma T}\;\eta + F$$

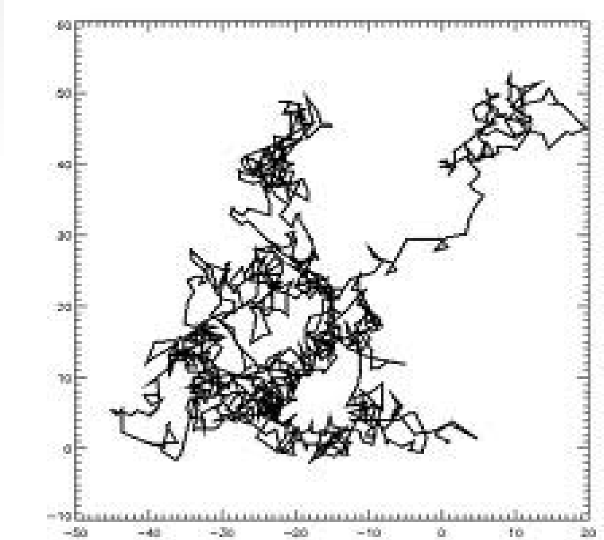
Colloidal particle pulled in an equilibrium fluid

Mean squared displacement in a perturbed state

$$\langle x^2(t)\rangle_F = 2D\;t + F^2(\beta D)^2\;t^2$$

DRIFT

$$\langle x(t)\rangle_F = \overline{\delta x(t)}_F = \mu F t$$



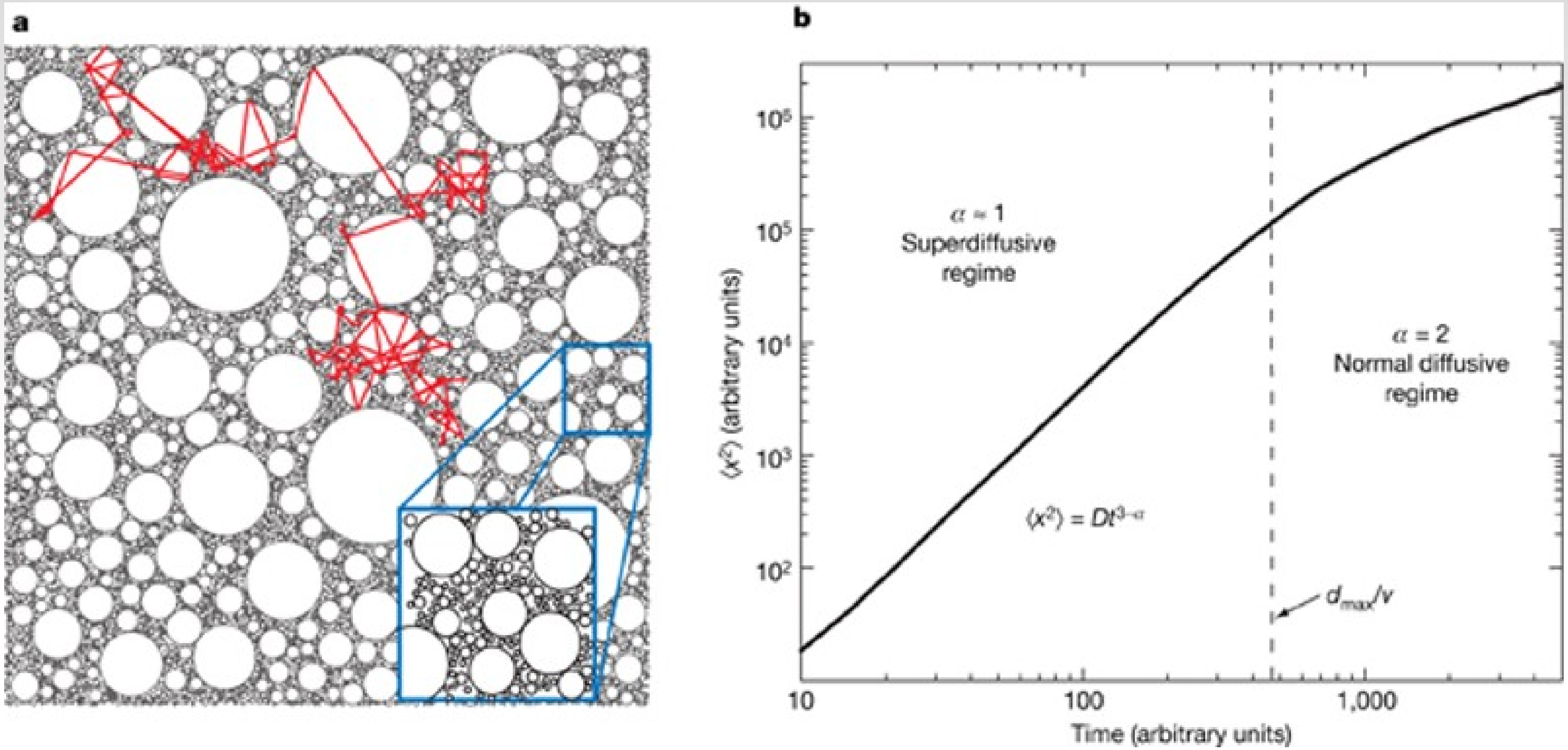
Einstein relation recovered by subtracting the squared drift

$$\frac{\langle x^2(t)\rangle_F - \langle x(t)\rangle_F^2}{\langle x(t)\rangle_{F+\delta F}} = \frac{2}{\beta F}$$

$$x_{i+1} = x_i + u$$

$$p(u) = \text{gaussian}$$

Levy Flights of Light



P. Barthelemy, J. Bertolotti, S. Wiersma, Nature, (2008)

$$x_{i+1} = x_i + u \qquad p(u) = \frac{1}{u^2} \qquad \langle x(t)^2 \rangle \sim t^2$$

Jumps of light rays

Continuous Time Random Walk (CTRW)

CTRW $$p(\Delta x, \Delta t) = p_x(\Delta x) p_t(\Delta t)$$

Sequence of pairwise random and
stochastically independent events

$$p_t(\Delta t) \sim \frac{1}{\Delta t^{1+\alpha}}$$

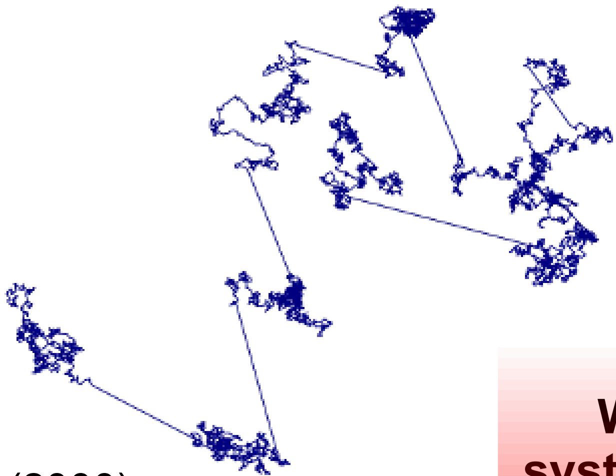
1) Particle stays at rest
for a time interval Δt

$$p_x(\Delta x) \sim \frac{1}{\Delta x^{1+\beta}}$$

2) Particle displays an
istantaneous jump Δx

$$\langle x^2(t) \rangle \sim t^{2\alpha/\beta}$$

$2\alpha > \beta$
Superdiffusion



$2\alpha < \beta$
Subdiffusion

R. Metzler, J. Klafter, Phys. Rep. 339 (2000)

What happens if we perturb a system with anomalous dynamics ?

LEVY WALK COLLISIONAL PROCESS

<u>**E. Barkai, V. N. Fleurov, PRE (1998)**</u>

1) **Probe** particle of mass **m** moves in a bath of scatterers with mass **M**

2) **Scatterers** are endowed with a velocity taken from a **gaussian** distribution p(V)

2) A trajectory is made of N+1 fligth times and N elastic scattering events

$$p(\tau) = \frac{1}{\tau^{1+\alpha}}$$

$$\{(\tau_1, V_1), (\tau_2, V_2), \ldots, (t_N, V_N), \tau_{N+1}\}$$

COLLISION

$$v_i = \gamma v_{i-1} + (1-\gamma) V_i$$

$$\gamma = \frac{\zeta - 1}{1 + \zeta}$$

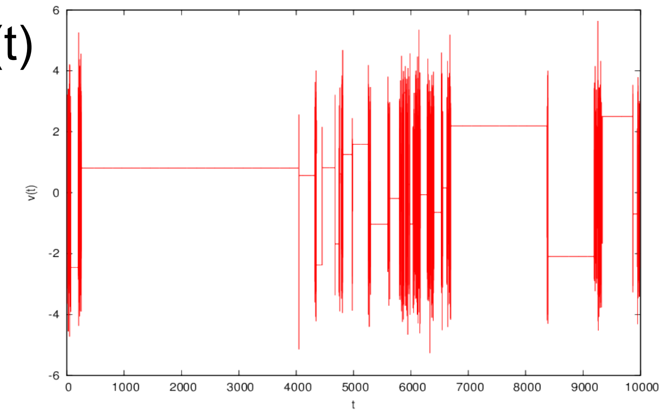
$$\zeta = m/M$$

TRAJECTORY

$$x(t) = \sum_i^N v_i \tau_i$$

$$\sum_i^N \tau_i = t$$

v(t)



Renewal process $v_i = V_i$

EINSTEIN RELATION AND SUPERDIFFUSION

Einstein relation holds at “equilibrium”: diffusion with no field is compared to drift with a field

$$\Delta x_i = v_i \tau_i + \frac{E}{2m}\tau_i^2$$

Constant accelaration during the flights

$$p(\tau) = \frac{1}{\tau^{1+\alpha}}$$

$$x(t) = \sum_i \Delta x_i$$

$$\langle x(t) \rangle_{\mathcal{E}} \sim \langle x^2(t) \rangle$$

GENERALIZED EINSTEIN RELATION FOR SUPERDIFFUSIVE DYNAMICS

$$\begin{cases} t^2 & \alpha < 1 & \langle \tau \rangle = \langle \tau^2 \rangle = \infty \\ t^{3-\alpha} & 1 < \alpha < 2 & \langle \tau \rangle < \infty \quad \langle \tau^2 \rangle = \infty \\ t & \alpha > 2 & \langle \tau \rangle < \infty \quad \langle \tau^2 \rangle < \infty \end{cases}$$

E. Barkai, V. N. Fleurov, PRE (1998)

MATCHING ARGUMENT FOR ASYMPTOTIC ESTIMATES

(0) ***Upper cutoff for in the power law distribution***

$$P_\tau(\tau) \sim \begin{cases} \tau^{-(1+\alpha)} & \text{if} \quad \tau < t_c \\ 0 & \text{if} \quad \tau > t_c \end{cases}$$

$$\langle \tau^n \rangle_c \sim t_c^{n-\alpha}$$

(1) $t \gg t_c$

$$\langle x^2(t) \rangle = \left\langle \left[\sum_{i=1}^{\overline{N(t)}} v_i \tau_i \right]^2 \right\rangle = \sum_{i=1}^{\overline{N(t)}} \langle v_i^2 \tau_i^2 \rangle + 2\overline{N(t)} \sum_{i=1}^{\overline{N(t)}} \langle v_i v_0 \tau_i \tau_0 \rangle$$

$$\langle x(t) \rangle_{\mathcal{E}} = \langle \sum_{i=1}^{N(t)} \left(v_i \tau_i + \frac{\mathcal{E}}{2} \tau_i^2 \right) \rangle = \frac{t}{\langle \tau \rangle_c} \left[\langle \tau \rangle_c \langle v \rangle + \frac{\mathcal{E}}{2} \langle \tau^2 \rangle_c \right] = \frac{\mathcal{E}}{2} \frac{t}{\langle \tau \rangle_c} \langle \tau^2 \rangle_c$$

$$\overline{N(t)} = \frac{t}{\langle \tau \rangle_c}$$

$$\langle x^2(t) \rangle = \frac{t}{\langle \tau \rangle_c} \langle v^2 \rangle \langle \tau^2 \rangle_c \qquad \langle x(t) \rangle_{\mathcal{E}} = \frac{\mathcal{E}}{2} \frac{t}{\langle \tau \rangle_c} \langle \tau^2 \rangle_c$$

(2) $t_c \to t$

$$\begin{cases} \alpha < 1 & \langle x^2(t) \rangle \sim t^2 & \langle \tau \rangle_c = t^{1-\alpha} \\ 1 < \alpha < 2 & \langle x^2(t) \rangle \sim t^{3-\alpha} & \langle \tau \rangle_c = \text{const} \end{cases}$$

G. Gradenigo, A. Sarracino, D. Villamaina, A. Vulpiani, *J. Stat. Phys.* (2012)

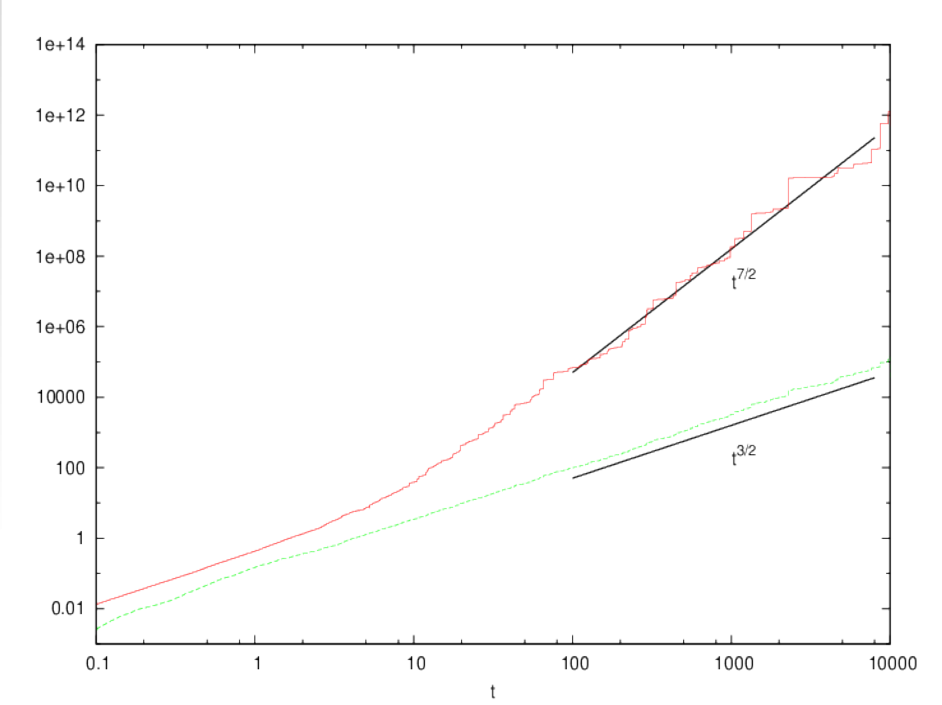
BREAKDOWN OF THE EINSTEIN RELATION IN PRESENCE OF A CURRENT

Fluctuations around the average position in presence of a current

$$\langle [\delta x(t)]^2 \rangle_{\mathcal{E}} \sim t^{7/2}$$

$$\langle x(t) \rangle_{\mathcal{E}} \sim t^{3/2}$$

BREAKDOWN OF THE EINSTEIN RELATION



$$\alpha = 3/2$$

$$p(\tau) = \frac{1}{\tau^{1+\alpha}}$$

$$\langle [\delta x(t)]^2 \rangle_{\mathcal{E}} = \langle x^2(t) \rangle_{\mathcal{E}} - \langle x(t) \rangle_{\mathcal{E}}^2 = t \left(\frac{\mathcal{E}^2}{4} \frac{\langle \tau^4 \rangle_c - \langle \tau^2 \rangle_c^2}{\langle \tau \rangle_c} + \frac{\langle v^2 \rangle \langle \tau^2 \rangle_c}{\langle \tau \rangle_c} \right)$$

$$1 < \alpha < 2 \qquad \langle x(t) \rangle_{\mathcal{E}} \sim t^{3-\alpha} \qquad \langle [\delta x(t)]^2 \rangle_{\mathcal{E}} \sim t^{5-\alpha}$$

BREAKDOWN OF THE EINSTEIN RELATION
FIELD INDUCED ANOMALOUS DYNAMICS

$$\langle [\delta x(t)]^2 \rangle_{\mathcal{E}} = \langle x^2(t) \rangle_{\mathcal{E}} - \langle x(t) \rangle_{\mathcal{E}}^2 = t \left(\frac{\mathcal{E}^2}{4} \frac{\langle \tau^4 \rangle_c - \langle \tau^2 \rangle_c^2}{\langle \tau \rangle_c} + \frac{\langle v^2 \rangle \langle \tau^2 \rangle_c}{\langle \tau \rangle_c} \right)$$

$$\langle x(t) \rangle_{\mathcal{E}} = \frac{\mathcal{E}}{2} \frac{t}{\langle \tau \rangle_c} \langle \tau^2 \rangle_c$$

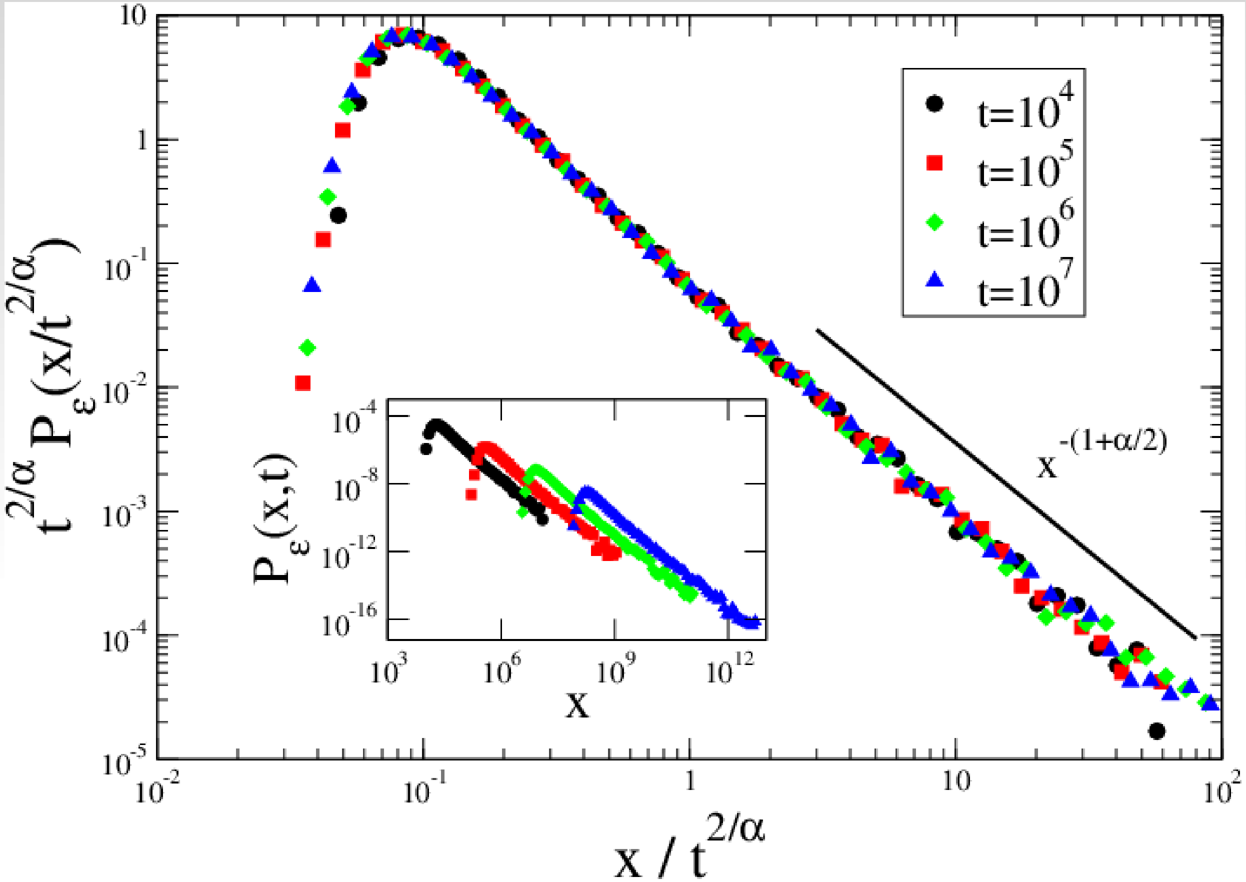
$$p(\tau) = \frac{1}{\tau^{1+\alpha}}$$

	$\langle x^2(t) \rangle$	$\langle x(t) \rangle_{\mathcal{E}}$	$\langle [\delta x(t)]^2 \rangle_{\mathcal{E}}$
$2 < \alpha < 4$	t	t	$t^{5-\alpha}$
$1 < \alpha < 2$	$t^{3-\alpha}$	$t^{3-\alpha}$	$t^{5-\alpha}$
$\alpha < 1$	t^2	t^2	t^4

$2 < \alpha < 4$ **Field induced anomalous dynamics**

G. Gradenigo, A. Sarracino, D. Villamaina, A. Vulpiani, *J. Stat. Phys.* (2012)

DISPLACEMENT DISTRIBUTION and RARE EVENTS



$1 < \alpha < 2$

$$p(\tau) = \frac{1}{\tau^{1+\alpha}}$$

$$\frac{1}{t^{2/\alpha}} F\left(\frac{x}{t^{2/\alpha}}\right)$$

(1) P(x,t) satisfies a scaling relation $\quad P_{\mathcal{E}}(x,t) \sim \frac{1}{t^\gamma} F\left(\frac{x}{t^\gamma}\right) \sim \frac{1}{t^\gamma}\left(\frac{x}{t^\gamma}\right)^{-(1+\alpha/2)}$

(2) Tail is ruled by isolated rare events $\quad P_{\mathcal{E}}(x,t) \sim \overline{N}(t) p(x) \sim \frac{t}{\langle t \rangle} \frac{1}{x^{1+\alpha/2}}$

$\gamma = 2/\alpha$

Long jumps $\quad x = v\ t + \mathcal{E}\ t^2/2 \sim t^2 \ \Rightarrow\ p(x) \sim 1/x^{-(1+\alpha/2)}$

WEAK VS STRONG ANOMALOUS DIFFUSION

P. Castiglione, A. Mazzino, P. Muratore-Gianneschi, A. Vulpiani (1998)

WEAK

One scaling length for P(x,t)

$$P(x,t) \sim \frac{1}{t^\gamma} F\left(\frac{x}{t^\gamma}\right) \qquad \ell(t) = t^\gamma$$

Behaviour of momenta is related to the scaling length

$$\langle x^n(t)\rangle \sim \int dx \, \frac{1}{t^\gamma} F\left(\frac{x}{t^\gamma}\right) \, x^n \sim t^{n\gamma} = \ell^n(t)$$

Levy Walk

$$1 < \alpha < 2$$

$$\left.\begin{aligned} &\frac{1}{t^{2/\alpha}} F\left(\frac{x}{t^{2/\alpha}}\right) \Rightarrow \ell(t) = t^{2/\alpha} \\ &\langle x^2(t)\rangle \sim t^{5-\alpha} \end{aligned}\right\}$$

$$\langle x^2(t)\rangle \neq \ell^2(t) = t^{4/\alpha}$$

STRONG

Several lengthscales for P(x,t)

$$\left.\begin{aligned} &P_\varepsilon(x,t) \sim \frac{1}{t^{2/\alpha}} F\left(\frac{x}{t^{2/\alpha}}\right) \Theta(t^2 - x) \\ &F(y) = y^{-(1+\alpha/2)} \end{aligned}\right\} \quad \langle x^2(t)\rangle \sim t^{5-\alpha}$$

$$\ell(t) = t^{2/\alpha}$$
$$\ell_{\text{cut}}(t) = t^2$$

FIELD-INDUCED ANOMALOUS DIFFUSION

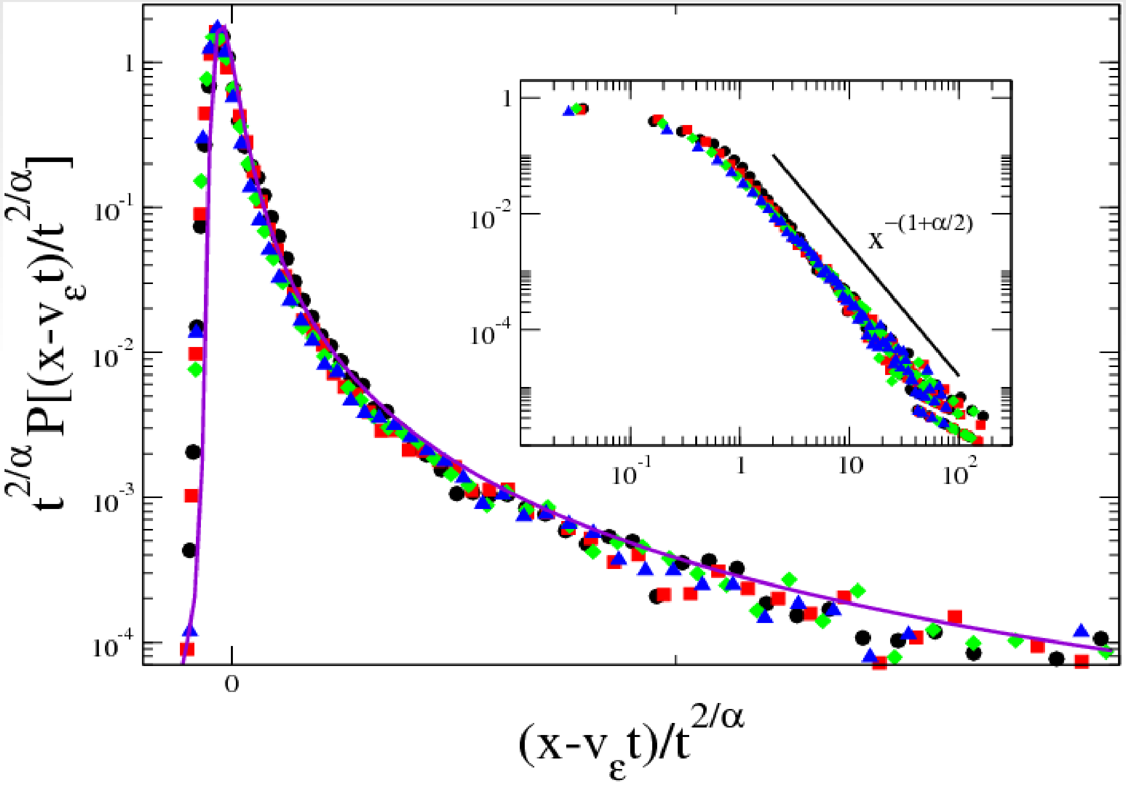
$2 < \alpha < 4$

$$P_{\mathcal{E}}(x,t) \sim \frac{1}{t^{2/\alpha}} F\left(\frac{x - v\ t}{t^{2/\alpha}}\right) \Theta(t^2 - x)$$

$$p(\tau) = \frac{1}{\tau^{1+\alpha}}$$

Probability distribution of displacements in single jump

$$x = vt + \frac{1}{2}\mathcal{E}t^2$$



$$x \sim t^2$$

$$\mathcal{E} > 0 \Rightarrow p(x) \sim \frac{1}{x^{1+\alpha/2}}$$

$$x \sim t$$

$$\mathcal{E} = 0 \Rightarrow p(x) \sim \frac{1}{x^{1+\alpha}}$$

When the field is switched off

$$P_{\mathcal{E}=0}(x,t) \sim \frac{1}{t^{1/2}} G\left(\frac{x - v\ t}{t^{1/2}}\right)$$

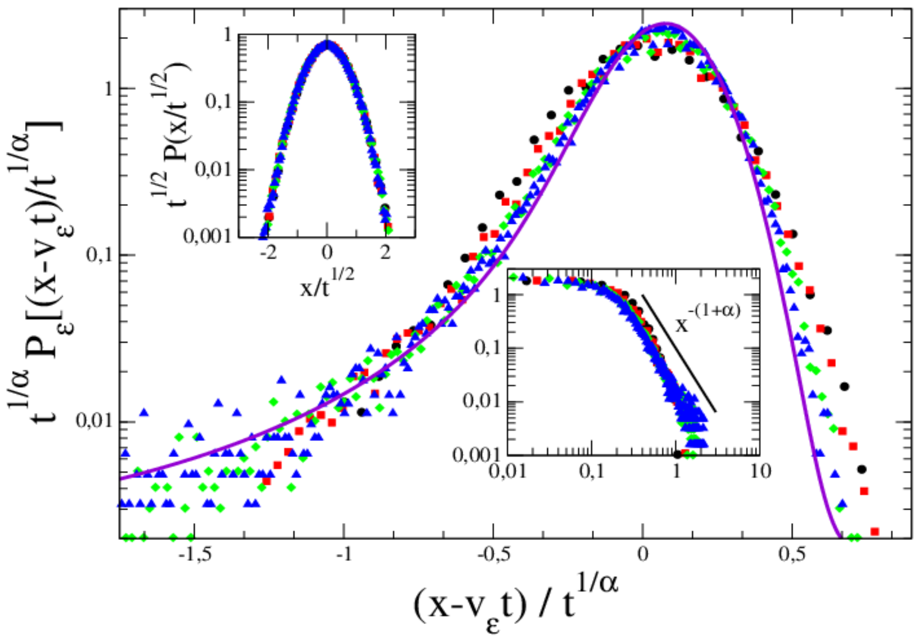
$$G(y) \sim e^{-x^2}$$

FIELD-INDUCED ANOMALOUS DIFFUSION
Continuous Time Random Walk WITH TRAPS

Process: Random Walk on a 1D lattice with a power law distribution of waiting times $p(\tau) = \frac{1}{\tau^{1+\alpha}}$

Field $p(x+\Delta x \mid x) = \frac{1}{2} + \epsilon \quad p(x-\Delta x \mid x) = \frac{1}{2} - \epsilon$

	$\langle x^2(t) \rangle$	$\langle x(t) \rangle_\mathcal{E}$	$\langle [\delta x(t)]^2 \rangle_\mathcal{E}$	
$1 < \alpha < 2$	t	t	$t^{3-\alpha}$	Field induced superdiffusion



$1 < \alpha < 2$

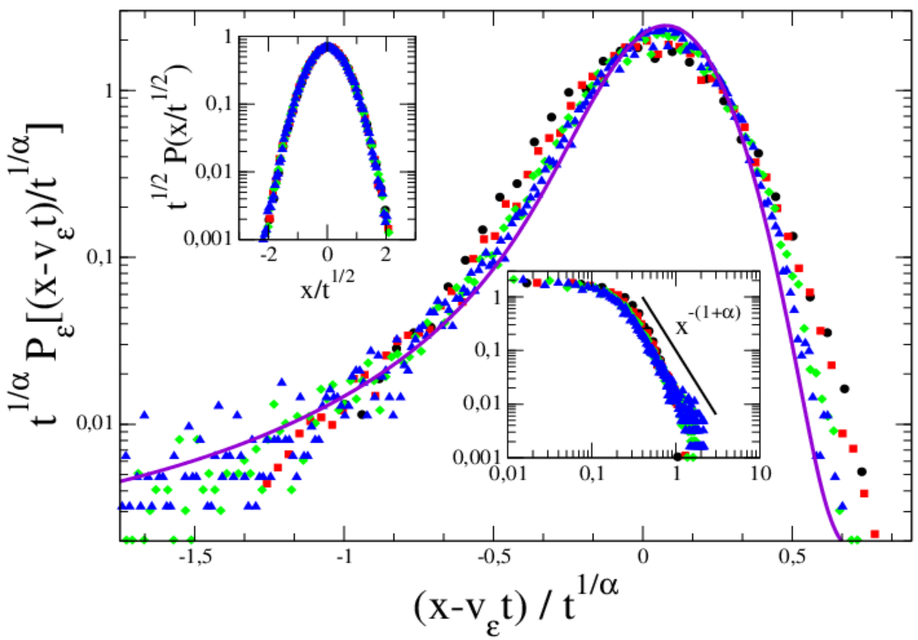
FIELD-INDUCED ANOMALOUS DIFFUSION
Continuous Time Random Walk WITH TRAPS

$1 < \alpha < 2$

Superdiffusive spreading

(1) Linear drift of the center of the distribution
(2) Particles trapped at the origin for a large time

$$P_\epsilon(\xi = vt, t) \sim \overline{N}(t)p(\xi) = \frac{t}{\langle t \rangle} \frac{1}{\xi^{1+\alpha}} \Theta(\xi)$$



R. Burioni, G. Gradenigo, A. Sarracino, A. Vezzani, A. Vulpiani, *J. Stat. Phys.* (2013)

CONCLUSIONS

The proportionanality between spontaneous fluctuations and drift, i.e. The Einstein relation, is “blind” to anomalous dynamics at “equilibrium”

The **Einstein relation is broken for anomalous dynamics only “out-of-equilibrium”**, namely when the perturbation is applied to a state which already has a finite current

We have underlined the importance of rare events for anomalous dynamics out of equilibrium

We have show scaling properties of the distribution function of displacements for driven anomalous dynamics

Einstein relation and subdiffusion

Perturbation on a state with ZERO current
Einstein relation at "equilibrium"

$$\overline{\delta x(t)}_F \sim \langle x^2(t)\rangle_0 \sim t^{2\nu} \qquad \nu < 1/2$$

R. Metzler, E. Barkai, J. Klafter, PRL (1999)
J-P. Bouchaud, A. Georges, Phys. Rep. (1990)

$$\frac{\langle x^2(t)\rangle}{\langle x(t)\rangle_F} = \frac{2}{\beta F}$$

Einstein relation holds !

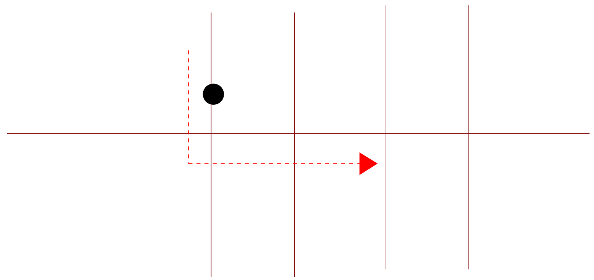
Perturbation on a state with <u>FINITE current</u>
<u>Einstein relation "out-of-equilibrium"</u>

Random walk on a comb: transition rates

$$\nu = 1/4$$

$$\langle x^2(t)\rangle_F \sim t^{1/2} \not\sim \overline{\delta x(t)}_{F+\delta F}$$

$$\langle x^2(t)\rangle_F - \langle x(t)\rangle_F^2 \not\sim \overline{\delta x(t)}_{F+\delta F}$$



$$\overline{\delta x(t)}_{F+\delta F} \sim \langle x^2(t)\rangle_F - \langle x(t) A(t,0)\rangle_F$$

A(t,0) = time spent on the backbone in [0,t]

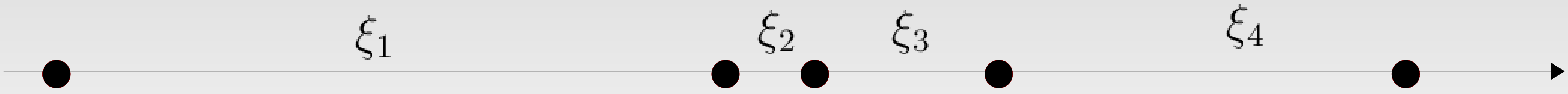
D.Villamaina, A.Sarracino, G.Gradenigo, A Puglisi, A Vulpiani, J.Stat.Mech (2011)

Levy Walk Collisional Process 1D

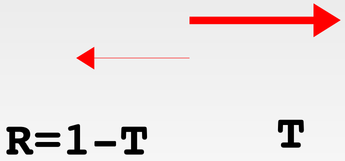
and the Levy-Lorentz gas

LEVY-LORENTZ GAS 1D

$$p(\xi) = \frac{1}{\xi^{\beta+1}}$$



Probe particle moves along
a line with velocity |v|=1

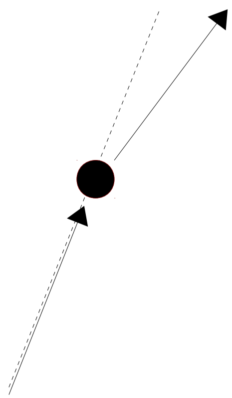


Non negligible correlations !
Finite reflection probability

E. Barkai, V. Fleurov, J. Klafter PRE (2000)
R. Burioni, L. Caniparoli, A. Vezzani PRE (2010)

Levy Walk Collisional Process 1D ~ LEVY LORENTZ GAS 2D

Randomness in postcollisional velocities in 1D
=
Randomness in scattering angles in 2D



If we also assumed an almost costant velocity modulus, peaked P(V) ...

$$p(\tau) = \frac{1}{\tau^{1+\alpha}} \longrightarrow p(r) = \frac{1}{r^{1+\alpha}}$$

Negligible probability of scattering angle 180°: **no correlations !**

Levy Walk Collisional process 1D

(Argument for asymptotic estimates)

Consider an upper cutoff for the power law distribution

$$P_\tau(\tau) \sim \begin{cases} \tau^{-(1+\alpha)} & \text{if} \quad \tau < t_c \\ 0 & \text{if} \quad \tau > t_c \end{cases}$$

DRIFT WITH FIELD AND UNBIASED NOISE

$$\langle x(t)\rangle_{\mathcal{E}} = \langle \sum_{i=1}^{N(t)} \left(v_i \tau_i + \frac{\mathcal{E}}{2}\tau_i^2 \right) \rangle = \frac{t}{\langle \tau \rangle_c} \left[\langle \tau \rangle_c \langle v \rangle + \frac{\mathcal{E}}{2} \langle \tau^2 \rangle_c \right] = \frac{\mathcal{E}}{2} \frac{t}{\langle \tau \rangle_c} \langle \tau^2 \rangle_c$$

MEAN SQUARE DISPLACEMENT

$$\langle x^2(t) \rangle = \frac{t}{\langle \tau \rangle_c} \langle v^2 \rangle \langle \tau^2 \rangle_c$$

$$\langle x(t)\rangle_{v_m} = \langle \sum_{i=1}^{N(t)} v_i \tau_i \rangle = \frac{t}{\langle \tau \rangle_c} \langle v \rangle_{v_m} \langle \tau \rangle_c = t \langle v \rangle_{v_m}$$

DRIFT WITH BIASED NOISE AND ZERO FIELD: ALWAYS LINEAR

BREAKING OF THE EINSTEIN RELATION FOR α<2

Levy Walk Collisional process 1D

Asymptotic estimates with correlated velocities

$$v_i = \gamma v_{i-1} + (1-\gamma) V_{i-1}$$

$\gamma \neq 0$

allows one to keep some memory across collisions

$$\langle x^2(t) \rangle = \sum_{i,j}^{N(t)} \langle v_i \tau_i v_j \tau_j \rangle \sim \frac{t}{\langle \tau \rangle_c} \left[\langle v^2 \rangle \langle \tau^2 \rangle_c + 2 \langle \tau \rangle_c^2 \sum_i^{N(t)} \langle v_i v_0 \rangle \right]$$

$t \lesssim t_c$ $\quad$ $\alpha < 1$ $\quad$ $t^{2-\alpha}$ > $t^{2-2\alpha}$

$$\langle x(t) \rangle_{\mathcal{E}} = \frac{\mathcal{E}}{2} \frac{t}{\langle \tau \rangle_c} \langle \tau^2 \rangle_c$$

EINSTEIN RELATION PRESERVED FOR ALL VALUES OF α WITH CORRELATED VELOCITIES

No appearing of velocities cross correlations in the drift !

α > 2 renormalization of diffusion coefficient

Levy Walk Collisional process 1D

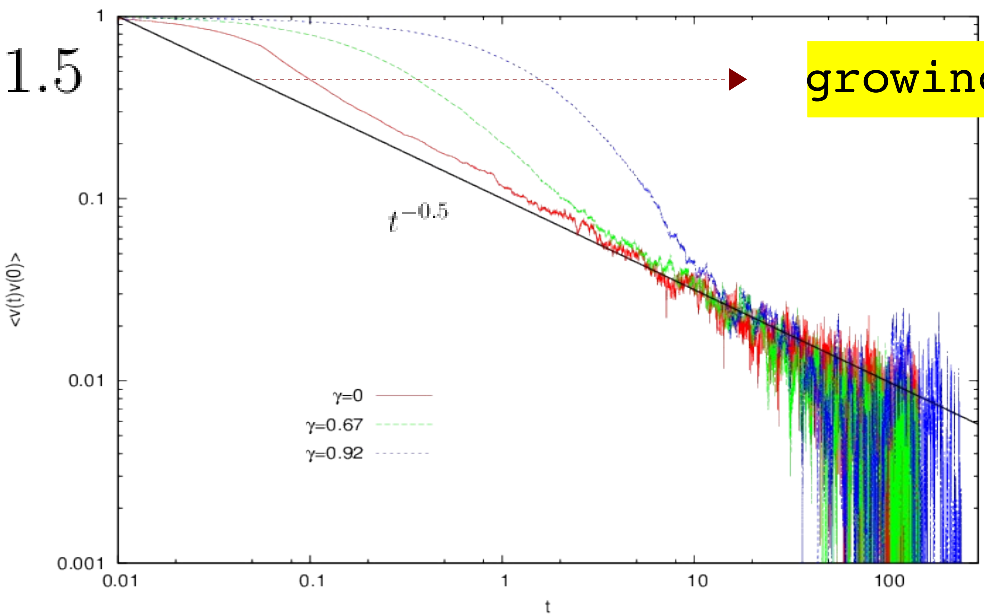
Velocity autocorrelation

$$\langle v(t)v(0)\rangle = \langle v(t)v(0)\rangle|_{n_c=0}\, P(n_c=0;[0,t]) + \langle v(t)v(0)\rangle|_{\overline{0}}\, P(\overline{0};[0,t])$$

Probability of having zero collisions starting the observation from an arbitrary time along the trajectory

$$\langle v(t)v(0)\rangle = \langle v^2\rangle \frac{t^{1-\alpha}}{\langle\tau\rangle} + e^{-t/\lambda(\gamma)}\left(1 - \frac{t^{1-\alpha}}{\langle\tau\rangle}\right)$$

$\alpha = 1.5$



growing γ

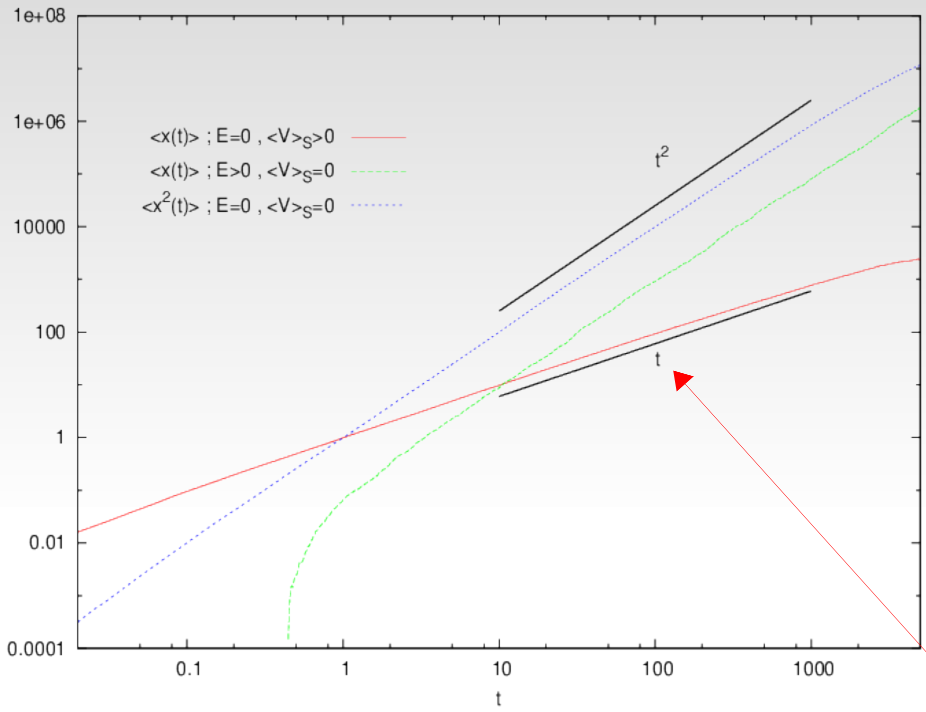
$$v_i = \gamma v_{i-1} + (1-\gamma)V_{i-1}$$

The increase of **γ** just increse characteristic time of the exponential

Power low tail of velocity correlation is only due to free flight events

Levy Walk Collisional Process 1D

(Differtent kind of perturbations)



$$p(\tau) = \frac{1}{\tau^{1+\alpha}}$$

$$\alpha = 1/2$$

$$\langle x^2(t) \rangle \sim t^2$$

$$E > 0$$

$$\langle x^2(t) \rangle \sim \overline{\delta x(t)}_F$$

Drift from external drag field
Einstein relation Ok

$$v_m > 0$$

$$\langle x^2(t) \rangle \nsim \overline{\delta x(t)}_F$$

Drift from biased noise
Breakdown of Einstein relation

Levy Walk Collisional process 1D

Perturbation of a state with a current

$$\langle[\delta x(t)]^2\rangle_{\mathcal{E}} = \langle x^2(t)\rangle_{\mathcal{E}} - \langle x(t)\rangle_{\mathcal{E}}^2 = t\left(\frac{\mathcal{E}^2}{4}\frac{\langle\tau^4\rangle_c - \langle\tau^2\rangle_c^2}{\langle\tau\rangle_c} + \frac{\langle v^2\rangle\langle\tau^2\rangle_c}{\langle\tau\rangle_c}\right)$$

		$E > 0$	
	$\langle x^2(t)\rangle$	$\langle x(t)\rangle_{\mathcal{E}}$	$\langle[\delta x(t)]^2\rangle_{\mathcal{E}}$
$2 < \alpha < 4$	t	t	$t^{5-\alpha}$
$1 < \alpha < 2$	$t^{3-\alpha}$	$t^{3-\alpha}$	$t^{5-\alpha}$
$\alpha < 1$	t^2	t^2	t^4

For 2<α<4 the breakdown of Einstein relation out of equilibrium shows an anomaly of the dynamics undetectable at equilibrium

Levy Walk Collisional process 1D

Perturbation of a state with a current

$$\langle [\delta x(t)]^2 \rangle_{\mathcal{E}} = \langle x^2(t) \rangle_{\mathcal{E}} - \langle x(t) \rangle_{\mathcal{E}}^2 = t \left(\frac{\mathcal{E}^2}{4} \frac{\langle \tau^4 \rangle_c - \langle \tau^2 \rangle_c^2}{\langle \tau \rangle_c} + \frac{\langle v^2 \rangle \langle \tau^2 \rangle_c}{\langle \tau \rangle_c} \right)$$

		$E > 0$	
	$\langle x^2(t) \rangle$	$\langle x(t) \rangle_{\mathcal{E}}$	$\langle [\delta x(t)]^2 \rangle_{\mathcal{E}}$
$2 < \alpha < 4$	t	t	$t^{5-\alpha}$
$1 < \alpha < 2$	$t^{3-\alpha}$	$t^{3-\alpha}$	$t^{5-\alpha}$
$\alpha < 1$	t^2	t^2	t^4

For 2<α<4 the breakdown of Einstein relation out of equilibrium shows an anomaly of the dynamics undetectable at equilibrium

Levy Walk Collisional process 1D

Perturbation of a state with a current

$$\langle[\delta x(t)]^2\rangle_{\mathcal{E}} = \langle x^2(t)\rangle_{\mathcal{E}} - \langle x(t)\rangle_{\mathcal{E}}^2 = t\left(\frac{\mathcal{E}^2}{4}\frac{\langle\tau^4\rangle_c - \langle\tau^2\rangle_c^2}{\langle\tau\rangle_c} + \frac{\langle v^2\rangle\langle\tau^2\rangle_c}{\langle\tau\rangle_c}\right)$$

		$E > 0$		$v_m > 0$	
	$\langle x^2(t)\rangle$	$\langle x(t)\rangle_{\mathcal{E}}$	$\langle[\delta x(t)]^2\rangle_{\mathcal{E}}$	$\langle x(t)\rangle_{v_m}$	$\langle[\delta x(t)]^2\rangle_{v_m}$
$2 < \alpha < 4$	t	t	$t^{5-\alpha}$	t	t
$1 < \alpha < 2$	$t^{3-\alpha}$	$t^{3-\alpha}$	$t^{5-\alpha}$	t	$t^{3-\alpha}$
$\alpha < 1$	t^2	t^2	t^4	t	t^2

For 2<α<4 the breakdown of Einstein relation out of equilibrium shows an anomaly of the dynamics undetectable at equilibrium