Edwards Thermodynamics for a Driven Athermal System with Dry Friction

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We obtain, using semianalytical transfer operator techniques, the Edwards thermodynamics of a one-dimensional model of blocks connected by harmonic springs and subjected to dry friction. The theory is able to reproduce the linear divergence of the correlation length as a function of energy density observed in direct numerical simulations of the model under tapping dynamics. We further characterize analytically this divergence using a Gaussian approximation for the distribution of mechanically stable configurations, and show that it is related to the existence of a peculiar infinite temperature critical point.

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Although systems governed by dissipative interactions do not obey equilibrium statistical mechanics, there have been several attempts to describe such systems with *effective* equilibriumlike theories. A paradigmatic example is the problem of amorphous packings of frictional grains, for which an effective thermodynamic was proposed by Edwards and co-workers [1–5]. This approach relies on the basic assumption that all mechanically stable packings of grains occupying the same volume have the same probability. This is expected if the system is repeatedly perturbed with "extensive operations" [3], like a shaking of the grains followed by a fast relaxation to a blocked (mechanically stable) configuration. One can then build an effective thermodynamics by determining all mechanically stable configurations (MSCs) of the grains, and computing the mean values of physical observables from flat averages over accessible blocked configurations. The predicted mean values can then be compared to dynamical averages obtained from a given "tapping" protocol that samples blocked configurations.

For athermal systems in which an energy is defined, Edwards' prescription can be formulated as follows [6]. One postulates the existence of an effective temperature $T_{\mathrm{Ed}} = \beta_{\mathrm{Ed}}^{-1}$ such that the probability of a blocked configuration $\mathcal{C}$ of energy $E(\mathcal{C})$ takes the form

$$P(\mathcal{C}) = \frac{1}{\mathcal{Z}} e^{-\beta_{\mathrm{Ed}} E(\mathcal{C})} \mathcal{F}(\mathcal{C}), \tag{1}$$

where $\mathcal{Z}$ is a generalized partition function; $\mathcal{F}(\mathcal{C}) = 1$ if $\mathcal{C}$ is a MSC and $\mathcal{F}(\mathcal{C}) = 0$ otherwise (only blocked configurations have a nonzero probability). This constraint is non-Hamiltonian, in the sense that it gives a zero probability to (mechanically unstable) configurations having a finite energy, whereas they would have a finite probability in canonical equilibrium. At first sight, Eq. (1) looks like a harmless generalization of equilibrium statistical mechanics, by simply restricting the set of accessible configurations. For instance, introducing an upper bound $|x_i| < X_{\max}$ on harmonic oscillators x_i does not deeply affect their statistical properties. However, the nontrivial point is that the constraints arising from MSCs are often much more complex than simply introducing a bound on individual variables; in particular, the constraint of mechanical stability may itself introduce strong correlations in the system. Notice that, beyond volume and energy, one may also take into account other quantities when building an Edwards-type thermodynamics [7–11] (e.g., the stress tensor).

Several attempts have been made to test the Edwards scenario, not only in packings of grains, experimentally [12–15] and numerically [16–20], but also in abstract models like spin systems and lattice gases [21–26], and in glass and spin-glass models [6,27–30]. Typically, one uses a specific tapping protocol to sample blocked states, and compares the dynamical average of the observables to the thermodynamic averages obtained from Eq. (1). Although it has been shown explicitly in some cases that the Edwards approach is not exact [25,26], or that fluctuations in specific blocked (or absorbing) states scale differently from what would be predicted by a thermodynamic approach [31], Edwards thermodynamics is generally believed to be a reasonably good description in many cases [5]. The main difficulty with the Edwards measure Eq. (1) is that the partition function $\mathcal{Z}$, from which all thermodynamic quantities can be derived, is very complicated to compute due to the complexity of the function $\mathcal{F}(\mathcal{C})$ characterizing blocked states [32–36]. Standard approaches are then either to consider abstract models [21–26], which are far from any realistic system but simple enough to allow for an explicit solution, or to resort to mean-field [37] or more involved [32] approximations, which still capture part of the interesting phenomenology, but (at least partly) miss relevant information about spatial correlations in the system.

In this Letter, we introduce a realistic model in which Edwards thermodynamics can be computed exactly. We investigate a one-dimensional model of frictional blocks connected by harmonic springs, subjected to a tapping

dynamics. Because of the one-dimensional geometry, statistical properties can be computed semianalytically in the thermodynamic limit using a transfer operator method. Our numerical simulation and theoretical results lead both to an infinite temperature critical point, with a correlation length diverging linearly with the stored energy density—directly measurable quantities in numerical simulations. We analytically confirm these results using a Gaussian approximation for the joint probability distribution of spring elongations, and further characterize this critical point in terms of the divergence of energy and length fluctuations.

Simulations.—Our model is represented by a one-dimensional chain of blocks of mass m connected by N harmonic springs sliding on a horizontal plane [38–42]. Each particle is subjected to dry (Coulomb) friction. The position of the ith mass is denoted as x_i. When a block is sliding it is subjected to a dissipative force proportional to the dynamic friction coefficient, $f_{i,\mathrm{diss}} = -\mu_d mg\,\mathrm{sgn}(\dot{x}_i)$, with g the gravitational constant. When at rest, it starts moving when the applied force exceeds the static friction force, $|f_i| > mg\mu_s$. We fix μ_s to a constant value both because we regard the blocks as macroscopic compared to the asperities of the surface, and because it makes the problem more tractable analytically. The elongation of the ith spring is $\xi_i = x_i - x_{i-1} - l_0$, with l_0 the constant rest length, so that the elastic force on each block reads $k(\xi_{i+1} - \xi_i)$, with k the spring stiffness. Taking into account also an external force f_i^{ext}, we have the following list of dimensionless variables: $\tilde{t} = t/\tau_0$, $\tilde{x} = x/(g\tau_0^2)$, $\tilde{f}_i^{\mathrm{ext}} = f_i^{\mathrm{ext}}/(mg)$, and $\tilde{l}_0 = l_0/(g\tau_0^2)$, with $\tau_0 = \sqrt{k/m}$. Dropping the tildes, the dimensionless equation of motion reads

$$\ddot{x}_i = -\mu_d \mathrm{sgn}(\dot{x}_i) + x_{i+1} + x_{i-1} - 2x_i + f_i^{\mathrm{ext}}, \qquad (2)$$

with $|\xi_{i+1} - \xi_i + f_i^{\mathrm{ext}}| > \mu_s$ the condition to start motion. We simulated a chain of $N + 1 = 256$ blocks with open boundary conditions—see the Supplemental Material [43]. No finite size effects appeared, changing the size from 64 to 2048. The "blocked" configurations are those which, in the absence of external force, are mechanically stable: $\forall i$, $\dot{x}_i = 0$, and $|\xi_{i+1} - \xi_i| < \mu_s$. We then define the following *tapping* dynamics: the external forces f_i^{ext} are switched on in Eq. (2) and act during a given period of time τ, after which they are switched off and the system relaxes to a MSC. This procedure, which we call the *driving cycle*, is repeated a large number of times to sample MSCs. At each cycle, the forces f_i^{ext} are drawn (randomly for each site i) from a distribution

$$p(f_i^{\mathrm{ext}}) = (1-\rho)\delta(f_i^{\mathrm{ext}}) + \frac{\rho}{\sqrt{2\pi\sigma^2}} e^{-(f_i^{\mathrm{ext}}-F)^2/2\sigma^2}. \qquad (3)$$

A driving protocol is determined by fixing the parameters ρ and σ. We studied two situations: pulling all blocks with a random force ($\rho = 1$, $\sigma > 0$) or pulling a fraction of them with a constant force ($\rho < 1$, $\sigma = 0$); see Fig. 1. This "disordered" driving protocol is aimed at sampling efficiently the MSCs. For a given protocol, one can then vary the intensity F and duration τ of the driving. Each MSC is characterized by the typical value of the energy stored by the springs $e = (1/2N)\sum_{i=1}^{N} \xi_i^2$. For each tapping protocol, the average energy $e(F,\tau)$ of the MSCs is found to depend only on the product $F\tau$ (see the Supplemental Material [43]).

To characterize the MCSs we focus on the correlation function $C(|i-j|) = \langle \xi_i \xi_j \rangle$ between the elongations of the springs at position i and j in the chain. Since this function is trivial [$C(|i-j|) = C_0\delta_{ij}$] in a thermal harmonic chain at all temperatures, any appearance of correlations is a signature of the unusual statistics associated with the non-Hamiltonian constraints. Correlation functions $C(|i-j|)$ measured for different tapping protocols are shown in Fig. 1. For a given tapping protocol we find that the extent of correlations *increases* when the average energy of the MSC increases. We extract the correlation length $\ell(e)$ for each case as the distance (i.e., number of springs) $|i-j|$ at which the measured correlation function decays below a conventional threshold $C^* = 0.2$. The insets of Figs. 1(a), 1(b), and 1(c) show the collapse of the correlation function when the x axis is rescaled with $\ell(e)$. Figure 1(d) shows,

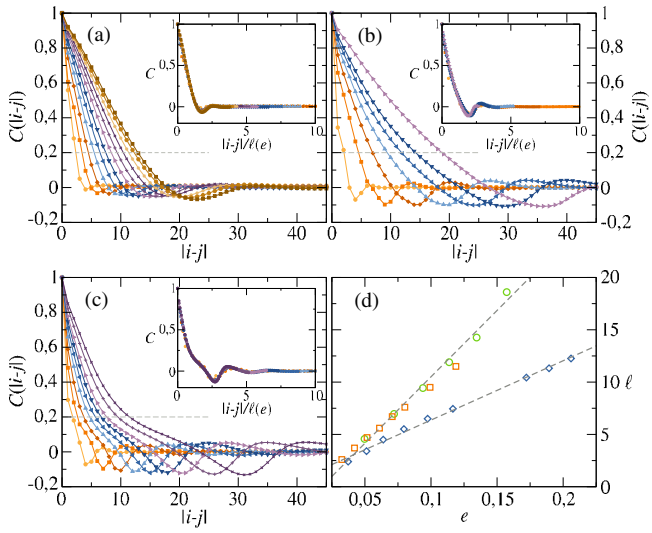

FIG. 1 (color online). (a),(b),(c) Correlation functions $C(|i-j|)$ for different tapping protocols, while in each panel the different curves correspond to different energy values. (a) $F \in [20, 128]$, $\rho = 0.3$, $\sigma = 0$; (b) $F \in [20, 140]$, $\rho = 0.8$, $\sigma = 0$; (c) $F \in [20, 128]$, $\rho = 1$, $\sigma = F/4$. $\tau = 60$ in all simulations. Inset $C(|i-j|)$ vs $|i-j|/\ell(e)$, showing a good collapse of the different curves. (d) Correlation length $\ell(e)$ as a function of the energy density e of the MSC, for the different tapping protocols shown in (a) diamonds, (b) circles, and (c) squares, showing a linear increase $\ell(e) \propto e$ with a protocol-dependent slope.

for all protocols studied, the correlation length growing linearly with the energy $\ell(e) \sim e$; the higher the energy, the more the system is correlated. This result may look counterintuitive, since commonly the correlation length decreases when energy increases. The key point is that for large ξ (a situation typical of high energy MCS) the frictional constraint $|\xi_{i+1} - \xi_i| < \mu_s$ imposes that ξ_{i+1} remains close to ξ_i. Therefore, correlations between spring extensions build up in the MSC. By comparing the correlation functions characterizing the blocked states and those at the end of the driving phase (when the force is switched off), we checked that correlations are not due to the external driving, but come entirely from static friction (see the Supplemental Material [43]). Despite the emergence of correlations, we find that the distribution of spring lengths is Gaussian at all energies; see Fig. 2.

Effective theory: Transfer operators.—Given that MSCs are defined in our model by the constraint $|\xi_{i+1} - \xi_i| < \mu_s$ $\forall i$, from Eq. (1) the probability of a configuration $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_N)$ reads

$$P(\boldsymbol{\xi}) = e^{-\beta_{\mathrm{Ed}} \sum_{i=1}^{N} \xi_i^2/2} \prod_{i=1}^{N} \Theta(\mu_s - |\xi_{i+1} - \xi_i|). \quad (4)$$

All the properties of the system can be obtained from the partition sum $\mathcal{Z} = \int_{-\infty}^{\infty} d\xi_1 \ldots d\xi_N P(\boldsymbol{\xi})$. Using the change of variables $\xi_i = \mu_s \xi_i'$, the partition function depends only (up to an irrelevant prefactor) on the product $\beta_{\mathrm{Ed}} \mu_s^2$; hence, all thermodynamic quantities are functions of T_{Ed}/μ_s^2. We consider periodic boundary conditions for the chain, without imposing any constraint on its total length. For convenience, we fix the rest length to $l_0 = \infty$, allowing us to take as the domain of integration $\xi_i \in (-\infty, \infty)$, while avoiding crossings of masses.

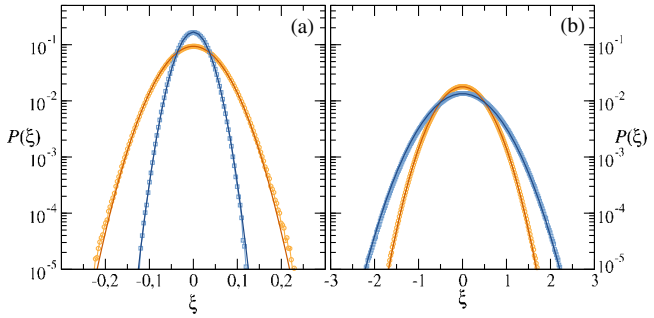

FIG. 2 (color online). Distributions of springs elongations $P(\xi)$ at different temperatures, on a semi-log scale. Points represent $P(\xi)$ in MSCs sampled via tapping, full lines represent $P(\xi)$ obtained from Edwards theory, computed at the temperature T_{Ed} yielding the same energy density e as the tapping. (a) Uncorrelated regime [$\ell(e) < 1$], where $e \sim T_{\mathrm{Ed}}$; $T_{\mathrm{Ed}} = 0.0008$ (blue squares) and 0.0026 (orange circles). (b) Correlated regime [$\ell(e) > 1$], where $e \sim \sqrt{T_{\mathrm{Ed}}}$; $T_{\mathrm{Ed}} = 1.19$ (orange circles) and 3.84 (blue squares).

Using Eq. (4), we have $\mathcal{Z} = \mathrm{Tr}(\mathcal{T}^N)$, with $\mathcal{T}$ an operator defined as $\mathcal{T}[f](x) = \int_{-\infty}^{\infty} dy T(x,y) f(x)$, being $T(x,y)$ the symmetric function:

$$T(x,y) = e^{-\beta_{\mathrm{Ed}} x^2/4} \Theta(\mu_s - |x - y|) e^{-\beta_{\mathrm{Ed}} y^2/4}. \quad (5)$$

The operator $\mathcal{T}$ has a maximum positive eigenvalue $\lambda_{\max}(\beta_{\mathrm{Ed}}, \mu_s)$, which can be computed numerically discretizing the domain of ξ, and using a complete orthonormal basis in L^2. All relevant thermodynamic observables are computed in the same way (see the Supplemental Material [43]). The free energy is obtained as $f = \beta_{\mathrm{Ed}}^{-1} \ln[\lambda_{\max}(\beta_{\mathrm{Ed}}, \mu_s)]$ while the energy reads $e = \partial(\beta_{\mathrm{Ed}} f)/\partial\beta_{\mathrm{Ed}} = -\langle \lambda_{\max} | \partial\mathcal{T}/\partial\beta_{\mathrm{Ed}} | \lambda_{\max} \rangle / \lambda_{\max}$. In the following, we compare results from theory and simulations by tuning the temperature T_{Ed} such that the energy e takes the same value as in the numerics.

The behavior of energy as a function of T_{Ed} from the transfer operator approach is shown in Fig. 3. We find two regimes separated by a crossover that depends on μ_s: for $T_{\mathrm{Ed}} \ll \mu_s^2$ there is an "equilibriumlike" regime where $e \sim T_{\mathrm{Ed}}$ while for $T_{\mathrm{Ed}} \gg \mu_s^2$ one finds $e \sim \sqrt{T_{\mathrm{Ed}}}$. The transfer operator approach allows us to compute also the probability distribution $p(\xi)$ of the elongation of a single spring (see the Supplemental Material [43]). The theoretical result for $p(\xi)$ is compared in Fig. 2 with the one estimated numerically from the MCSs, showing good agreement. We find that $p(\xi)$ is Gaussian in all regimes, even when correlations are present.

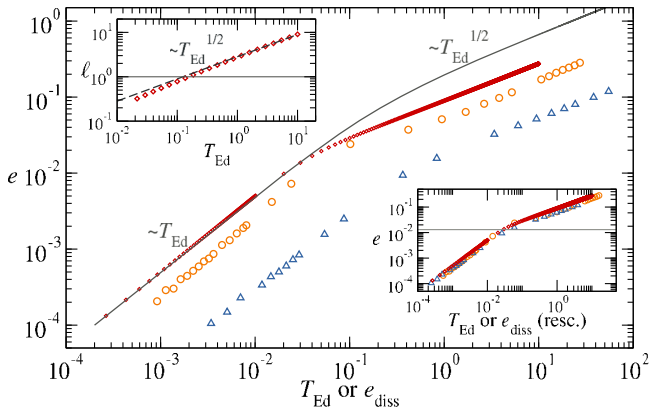

FIG. 3 (color online). Energy density e of MSC as a function of T_{Ed}, from transfer operators (small red diamonds) or Gaussian approximation (full line), and as a function of dissipated energy e_{diss} for two tapping protocols: (i) $\rho = 0.3$, $\sigma = 0$ (orange circles); (ii) $\rho = 1$, $\sigma > 0$ (blue triangles). For $T_{\mathrm{Ed}} \ll \mu^2$ one finds an equilibriumlike regime, $e \sim T_{\mathrm{Ed}}$; for $T_{\mathrm{Ed}} \gg \mu^2$ the behavior is $e \sim \sqrt{T_{\mathrm{Ed}}}$. Top inset: Correlation length from exact calculation (transfer operators); the behavior $\ell(T_{\mathrm{Ed}}) \sim \sqrt{T_{\mathrm{Ed}}} \sim e$ is clear when $T_{\mathrm{Ed}} \gg \mu^2$. Bottom inset: Same symbols, data sets of the main panel are collapsed by just rescaling the x axis; up to a protocol-dependent prefactor we have $e_{\mathrm{diss}} \sim T_{\mathrm{Ed}}$.

We also compute the correlation $C(|i-j|) = \langle \xi_i \xi_j \rangle$, which is very close to an exponential form for all values of T_{Ed}, $C(|i-j|) \propto e^{-|i-j|/\ell(\beta_{\mathrm{Ed}},\mu)}$ (see the Supplemental Material [43]). When $T_{\mathrm{Ed}} \gg \mu_s^2$ both the correlation length ℓ and the energy e grow as $\sqrt{T_{\mathrm{Ed}}}$ (see the inset of Fig. 3), implying $\ell \sim e$. We thus recover from Edwards thermodynamics the scaling behavior of the correlation length with energy observed in the simulated tapping dynamics. This is a remarkable success of the Edwards approach for this system. Conversely, there is almost no correlation between neighboring springs ($\ell < 1$), in the equilibrium-like low energy regime ($e \sim T_{\mathrm{Ed}}$).

We further show in the second inset of Fig. 3 that a direct measure of T_{Ed} (within a protocol dependent factor) is obtained from the dissipated energy per tapping cycle and particle, $e_{\mathrm{diss}} = \mu_d \langle \int_0^\tau \mathrm{sgn}[\dot{x}_i(t)] \dot{x}_i(t) dt \rangle$. Indeed, in the simulations e_{diss} is found to have the same scaling with e as the temperature obtained within the transfer operator approach (i.e., $e \sim e_{\mathrm{diss}}$ if $e \ll \mu_s^2$, $e \sim \sqrt{e_{\mathrm{diss}}}$ if $e \gg \mu_s^2$). The dissipated energy can therefore be interpreted as the analog of the thermal energy that allows the system to sample the configuration space [45].

Effective theory: Gaussian ansatz.—To obtain approximate analytical expressions for the thermodynamic quantities, we replace the Heaviside function in Eq. (4) by a Gaussian function,

$$\Theta(\mu_s - |\xi_{i+1} - \xi_i|) \to \frac{1}{\sqrt{\pi}} \exp\left(-\frac{|\xi_{i+1} - \xi_i|^2}{4\mu_s^2} \right), \quad (6)$$

yielding $\mathcal{Z} \propto \int \mathcal{D}\boldsymbol{\xi} e^{-S(\boldsymbol{\xi})}$ with an effective Hamiltonian

$$S(\boldsymbol{\xi}) = \frac{1}{2} \left[\beta_{\mathrm{Ed}} \sum_{i=1}^{N} \xi_i^2 + \frac{1}{2\mu_s^2} \sum_{i=1}^{N} (\xi_{i+1} - \xi_i)^2 \right]. \quad (7)$$

This effective Hamiltonian corresponds to a positive definite quadratic form $S(\boldsymbol{\xi}) = \frac{1}{2} \boldsymbol{\xi}^{\mathbf{T}} A \boldsymbol{\xi}$, where A is a symmetric real Toeplitz matrix (see the Supplemental Material [43]). The matrix A can be exactly diagonalized, yielding analytical expressions for energy, entropy, correlation function, and correlation length. The mean energy per particle reads

$$e(T_{\mathrm{Ed}}, \mu_s) = \frac{1}{2} \frac{\mu_s T_{\mathrm{Ed}}}{\sqrt{2T_{\mathrm{Ed}} + \mu_s^2}}, \quad (8)$$

from which we recover that the crossover point between the behaviors $e \sim T_{\mathrm{Ed}}$ and $e \sim \sqrt{T_{\mathrm{Ed}}}$ is $T_{\mathrm{Ed}}^* \approx \mu_s^2$. In Fig. 3, Eq. (8) is compared with the result from the transfer operator Eq. (5), showing a semiquantitative agreement. We also find that the correlation function is $\langle \xi_i \xi_j \rangle \sim e^{-|i-j|/\ell(T_{\mathrm{Ed}},\mu_s)}$ with $\ell(T_{\mathrm{Ed}}, \mu_s)$ such that $\ell \sim \sqrt{T_{\mathrm{Ed}}}/\mu_s$ in the limit $T_{\mathrm{Ed}} \gg \mu_s^2$. This result can be recovered from a field-theoretic viewpoint, by taking a continuous limit in Eq. (7), yielding

$$S[\xi] \propto \int dx \left[\frac{1}{2} \left(\frac{\partial \xi}{\partial x} \right)^2 + \frac{1}{2} m^2 \xi^2(x) \right] \quad (9)$$

with a "mass" term $m^2 = 2\mu_s^2 \beta_{\mathrm{Ed}}$. The correlation function of such a Gaussian field theory is known [46] to be $\langle \xi(x) \xi(y) \rangle \sim e^{-m|x-y|}$, so that we recover a correlation length $\ell \sim \beta_{\mathrm{Ed}}^{-1/2}$. This field-theoretic formulation confirms the presence of an infinite temperature critical point, since the mass term goes to zero at infinite temperature. To inspect the critical exponents associated with this critical point, we study the fluctuations of the total energy of the chain, $\delta E = \frac{1}{2} \sum_{i=1}^{N} (\xi_i^2 - \langle \xi_i^2 \rangle)$ and the fluctuations of its total length, $\delta \mathcal{L} = \sum_i \xi_i$. We find (see the Supplemental Material [43]) that the variance of both energy and length diverge linearly with temperature (or, equivalently, as ℓ^2),

$$\frac{\langle [\delta E]^2 \rangle}{N} \sim \frac{\langle [\delta \mathcal{L}]^2 \rangle}{N} \sim T_{\mathrm{Ed}}. \quad (10)$$

Finally, we compute the entropy density $s = -\partial f / \partial T_{\mathrm{Ed}}$. We find that it saturates at high temperature to a finite value, $\lim_{T_{\mathrm{Ed}} \to \infty} s(T_{\mathrm{Ed}}, \mu_s) = \frac{1}{2} \ln 2 + \ln \mu_s$. This saturation results from the presence of long-range correlation at infinite temperature. This can be confirmed by contrast, computing the "mean-field" entropy density $s_{\mathrm{mf}} = -\int d\xi p(\xi) \ln p(\xi)$, with $p(\xi)$ the distribution of a single spring elongation ξ. We find that s_{mf}, which discards correlations, diverges like $\ln T_{\mathrm{Ed}}$ at infinite temperature (see the Supplemental Material [43]), at odds with the saturation of the entropy s.

Conclusions.—The present study provides a clear-cut example of how an effective thermodynamic theory can successfully describe an *athermal* dissipative system. We believe that an important ingredient for the Edwards theory to hold is that the entropy of blocked states is extensive, as it holds in our case. The most remarkable difference between standard equilibrium thermodynamics and the effective theory we have presented is the presence of an infinite temperature critical point, with an associated divergence of the correlation length as $\ell \sim T_{\mathrm{Ed}}^{1/2}$ (or $\ell \sim e$). As seen in the field-theoretic formulation of Eq. (9), this infinite temperature critical point results from the long-range correlation generated by static friction in the blocked states. The difference with standard equilibrium systems is that the gradient term in the effective Hamiltonian does not come from an energetic interaction, but from a non-Hamiltonian constraint. Its coefficient is strictly temperature independent, while the coefficient of the energetic term scales inversely with temperature. While temperature-independent terms could also be present at equilibrium (e.g., entropic constraints such as excluded volume), they are usually purely local and do not involve gradient terms. Hence, in spite of its simplicity, our model exhibits a phenomenology clearly distinct from that of equilibrium systems, and the field-theoretic formulation

suggests that the results should be quite robust to changes in the details of the model. Future work should investigate this issue in more detail.

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